# SADLER MATHEMATICS METHODS UNIT 3

## WORKED SOLUTIONS

## Chapter 5 The fundamental theorem of calculus

### Exercise 5A

**Question 1**

**a** $\int (12t^2 + 6t)\,dt$

$= 4t^3 + 3t^2 + c$

**b** $\int_1^x (12t^2 + 6t)\,dt$

$= \left[4t^3 + 3t^2\right]_1^x$

$= 4x^3 + 3x^2 - (4 + 3)$

$= 4x^3 + 3x^2 - 7$

**c** $\frac{d}{dx}\left(\int_1^x (12t^2 + 6t)\,dt\right)$

$= \frac{d}{dx}\left(4x^3 + 3t^2 - 7\right)$

$= 12x^2 + 6x$

**Question 2**

**a**

$$\int (1 - t^{-2})dt$$

$$= t - \frac{t^{-1}}{-1} + c$$

$$= t + \frac{1}{t} + c$$

**b**

$$\int_3^x (1 - \frac{1}{t^2})\ dt$$

$$= \left[ t + \frac{1}{t} \right]_3^x$$

$$= (x + \frac{1}{x}) - (3 + \frac{1}{3})$$

$$= x + \frac{1}{x} - \frac{10}{3}$$

**c**

$$\frac{d}{dx}\left( \int_3^x (1 + \frac{1}{t^2})dt \right)$$

$$= \frac{d}{dx}\left( x + \frac{1}{x} - \frac{10}{3} \right)$$

$$= 1 - \frac{1}{x^2}$$

## Question 3

**a** $\int 2t(t^2+3)^4\,dt$

$= \dfrac{(t^2+3)^5}{5} + c$

**b** $\int_{-2}^{x} 2t(t^2+3)^4\,dt$

$= \left[\dfrac{(t^2+3)^5}{5}\right]_{-2}^{x}$

$= \dfrac{(x^2+3)^5}{5} - \dfrac{((-2)^2+3)^5}{5}$

$= \dfrac{(x^2+3)}{5} - \dfrac{16\,807}{5}$

**c** $\dfrac{d}{dx}\left[\int_{-2}^{x} 2t(t^2+3)^4\,dt\right]$

$= \dfrac{d}{dx}\left(\dfrac{(x^2+3)^5}{5} - \dfrac{16\,807}{5}\right)$

$= \dfrac{5(x^2+3)^4 \times 2x}{5}$

$= 2x(x^2+3)^4$

## Question 4

$\dfrac{d}{dx}\left(\int_{a}^{x} f(t)dt\right) = f(x)$

## Question 5

$4x$

## Question 6

$5x^2$

## Question 7

$2x^3$

**Question 8**

$$\frac{2x}{5-x}$$

**Question 9**

$(x+3)^4$

**Question 10**

$16x(x^2+3)^4$

## Exercise 5B

It is hoped students will be able to recognise the relationship and produce the answer in one step, but just in case questions 1, 2 and 3 are done in full.

**Question 1**

$$\frac{d}{dx}\int_0^x \left(2t+3t^2\right)dt$$
$$=\frac{d}{dx}\left(\left[t^2+t^3\right]_0^x\right)$$
$$=\frac{d}{dx}\left((x^2+x^3)-0\right)$$
$$=2x+3x^2$$

**Question 2**

$$\frac{d}{dx}\left(\int_1^x (t^4+5)dt\right)$$
$$=\frac{d}{dx}\left(\left[\frac{t^5}{5}+5t\right]_1^x\right)$$
$$=\frac{d}{dx}\left(\frac{x^5}{5}+5x-(\frac{1}{5}+5)\right)$$
$$=x^4+5$$

**Question 3**

**a**

$$\int \frac{d}{dx}(x^2+5)$$
$$\int 2x$$
$$=x^2+c$$

**b**

$$\int \frac{d}{dx}\left(6x^3-4x^2+2x+1\right)dx$$
$$=\int\left(18x^2-8x+2\right)dx$$
$$=6x^3-4x^2+2x+c$$

**Question 4**

**a** $f(3)=1$

**b** $f(6)=-2$

**c** $\int_4^0 f(x)\,dx$

$=\frac{1}{2}\times 4\times 4$

$=8$

**d** $\int_0^{10} f(x)\,dx$

$=8-8-\frac{1}{2}\times 2\times 2$

$=-2$

**Question 5**

**a** $f(1)=-2$

**b** $f(7)=3$

**c** $\frac{1}{2}(2+3)\times 2=5$

**d** Area from $x=1$ to $x=3$

$2+\frac{1}{2}\times 1\times 2=3$

Area from $x=3$ to $x=8$

$6+5+1=12$

$\therefore \int_1^8 f(x)\,dx=12-3=9$

## Question 6

**a** $\int_0^2 f(x)\,dx$

$= \frac{1}{4} \times \pi \times 2^2$

$= \pi$

**b** $\int_2^{10} f(x)\,dx$

$= 4 - \pi$

$\int_6^8 = -\int_2^4 = -\int_4^6 \quad \therefore \text{Area} = 4 + (-\pi)$

**c** $\int_0^a f(x)\,dx = 0$

When $a = 0$, 4 and 8

**d** $\int_0^a f(x)\,dx < 0$

When $4 < a < 8$

## Question 7

$y = x^{\frac{5}{2}} + \int_0^x (1+3t^2)^4\,dt$

$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + (1+3x^2)^4$

## Miscellaneous exercise five

**Question 1**

$$6x^2 - \frac{1}{2}x^{-\frac{1}{2}} = 6x^2 + \frac{1}{2\sqrt{x}}$$

**Question 2**

$$(x+5)(x-3) = x^2 + 2x - 15$$

$$\frac{d}{dx}(x^2 + 2x - 15)$$

$$= 2x + 2$$

**Question 3**

$$2(3x-1) \times 3 = 6(3x-1)$$

**Question 4**

$$5(3x-1)^4 \times 3 = 15(3x-1)^4$$

**Question 5**

$$(5x-1)(6x^2) + (2x^3 - 3)(5)$$

$$= 30x^3 - 6x^2 + 10x^3 - 15$$

$$= 40x^3 - 6x^2 - 15$$

**Question 6**

$$(5x-1) \times 3(2x-3)^2 \times 2 + (2x-3)^3(5)$$

$$= (2x-3)^2\left[6(5x-1) + 5(2x-3)\right]$$

$$= (2x-3)^2\left[30x - 6 + 10x - 15\right]$$

$$= (2x-3)^2(40x-21)$$

**Question 7**

$$\frac{(x-1)\times 2-(2x+3)\times 1}{(x-1)^2}$$

$$=\frac{2x-2-2x-3}{(x-1)^2}$$

$$=-\frac{5}{(x-1)^2}$$

**Question 8**

$$\frac{(2x+3)\times 1-(x-1)\times 2}{(2x+3)^2}$$

$$=\frac{2x+3-2x+2}{(2x+3)^2}$$

$$=\frac{5}{(2x+3)^2}$$

**Question 9**

$$\frac{dy}{dx}=(x-1)(2x)+(x^2-2)(1)$$

$$=2x^2-2x+x^2-2$$

$$=3x^2-2x-2$$

When $x=0$,

$$\frac{dy}{dx}=-2$$

**Question 10**

$$y = x + \frac{6}{x}$$

$$\frac{dy}{dx} = 1 - \frac{6}{x^2} = 0$$

$$\frac{6}{x^2} = 1$$

$$x^2 = 6$$

$$x = \pm\sqrt{6}$$

When $x = +\sqrt{6}$,

$$y = \sqrt{6} + \frac{6}{\sqrt{6}}$$

$$= 2\sqrt{6}$$

When $x = -\sqrt{6}$,

$$y = -\sqrt{6} - \frac{6}{\sqrt{6}}$$

$$= -2\sqrt{6}$$

$$\frac{d^2y}{dx^2} = \frac{12}{x^3}$$

When $x = \sqrt{6}$,

$$\frac{d^2y}{dx^2} = \frac{12}{(\sqrt{6})^3} > 0$$

$\therefore (\sqrt{6},\ 2\sqrt{6})$ is a minimum point.

When $x = -\sqrt{6}$,

$$\frac{d^2y}{dx^2} = \frac{12}{(-\sqrt{6})^3} < 0$$

$\therefore (-\sqrt{6}, -2\sqrt{6})$ is a maximum point.

## Question 11

**a** The curve has a $y$-intercept (0, 3) and contains the point (5, 28).

The function $f(x)$ has a value of 3 when $x = 0$.

The function $f(x)$ has a value of 28 when $x = 5$.

The average rate of change of $f(x)$, from $x = 0$ to $x = 5$, is 5 units per unit change in $x$.

The instantaneous rate of change of $f(x)$ when $x = 1$ is 2.

**b**

$$f(x) = ax^2 + bx + c$$
$$f(0) = c = 3$$
$$f(5) = 25a + 5b + 3 = 28$$
$$25a + 5b = 25$$
$$f'(x) = 2ax + b$$

$$f'(1) = 2a + b = 2$$
$$b = 2 - 2a$$
$$25a + 5(2 - 2a) = 25$$
$$25a + 10 - 10a = 25$$
$$15a = 15$$
$$a = 1$$
$$b = 2 - 2(1)$$
$$= 0$$
$$\therefore f(x) = x^2 + 3$$

**c**

$$f(x) = ax^3 + bx^2 + cx + d$$
$$f(0) = d = 3$$
$$f(5) = 125a + 25b + 5c + 3 = 28$$
$$125a + 25b + 5c = 25$$
$$25a + 5b + c = 5 \quad \rightarrow \text{Equation 1}$$

$$f'(x) = 3ax^2 + 2bx + c$$
$$f'(1) = 3a + 2b + c = 2 \quad \rightarrow \text{Equation 2}$$

$$\text{Equation 1} - \text{Equation 2}$$
$$22a + 3b = 3$$

Select a pair of values for $a$ and $b$ which make $22a + 3b = 3$ true.

i.e. $a = 1.5, b = -10 \Rightarrow c = 17.5$

$$f(x) = 1.5x^3 - 10x^2 + 17.5x + 3$$

**Question 12**

$$y = 25\int 2(2x+1)^4 dx$$
$$= 25\times\frac{(2x+1)^5}{5}+c$$
$$= 5(2x+1)^5+c$$

When $x = 0,\ y = 7$

$$7 = 5(2(0)+1)^5+c$$
$$c = 2$$
$$\therefore y = 5(2x+1)^5+2$$

**Question 13**

$$f''(x) = 144(2x-1)^2$$
$$f'(x) = 72\int 2(2x-1)^2 dx$$
$$= 72\times\frac{(2x-1)^3}{3}+c$$
$$= 24(2x-1)^3+c$$

$$f'(1) = 24(2(1)-1)^3+c = 26$$
$$c = 2$$
$$f'(x) = 24(2x-1)^3+2$$
$$f(x) = \int 24(2x-1)^3+2\ dx$$
$$= \int 12\times 2(2x-1)^3+2\ dx$$
$$= 12\times\frac{(2x-1)^4}{4}+2x+c$$
$$= 3(2x-1)^4+2x+c$$

$$f(1) = 3(2(1)-1)^4+2(1)+c = 6$$
$$c = 1$$
$$f(x) = 3(2x-1)^4+2x+1$$

## Question 14

Marginal Revenue $= R'(x)$

$$R'(x) = 30 - 2(0.02)x$$
$$= 30 - 0.04x$$
$$R'(100) = 30 - 0.04(100)$$
$$= 26$$

Marginal revenue at $x = 100$ is \$26 per unit. The revenue will increase by approximately \$26.

## Question 15

**a** $y = (x+1)(x-2)^2$

Intercepts

$y$-intercept : $y = (0+1)(0-2)^2$

$$= 4$$
$$\therefore (0,\ 4)$$

$x$-intercept : $(x+1)(x-2)^2 = 0$

$$x = -1,\ 2$$

$\therefore$ It cuts the $x$-axis at $(-1,\ 0)$ and touches at $(2,\ 0)$.

Stationary points

$$\frac{dy}{dx} = (x+1)\times 2(x-2)\times 1 + (x-2)^2 \times 1$$
$$= 2(x+1)(x-2) + x^2 - 4x + 4$$
$$= 2(x^2 - x - 2) + x^2 - 4x + 4$$
$$= 3x^2 - 6x$$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$\therefore x = 0,\ x = 2$$

Stationary points at (0, 4) and (2, 0).

$$\frac{d^2y}{dx^2} = 6x - 6$$

When $x = 0$,

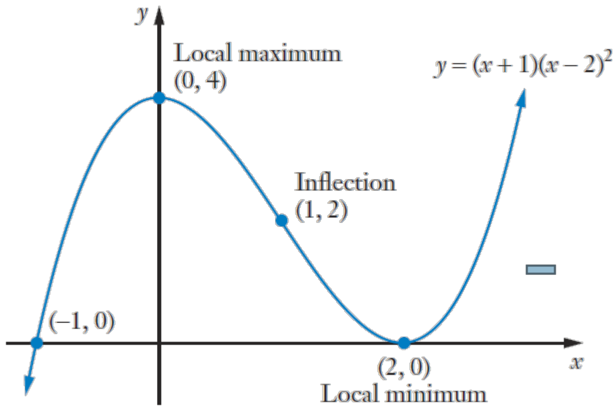
$$\frac{d^2y}{dx^2} = 6(0) - 6 < 0$$

$\therefore$ (0, 4) is a maximum point.

When $x = 2$,

$$\frac{d^2y}{dx^2} = 6(2) - 6 > 0$$

$\therefore$ (2, 0) is a minimum point.




**b** $\int_{-1}^{2} (x+1)(x-2)^2\, dx$

$$= \int_{-1}^{2} (x^3 - 3x^2 + 4)\, dx$$

$$= \left[\frac{x^4}{4} - x^3 + 4x\right]_{-1}^{2}$$

$$= \left[\frac{16}{4} - 8 + 8\right] - \left[\frac{1}{4} + 1 - 4\right]$$

$$= 4 - \left(-2\frac{3}{4}\right)$$

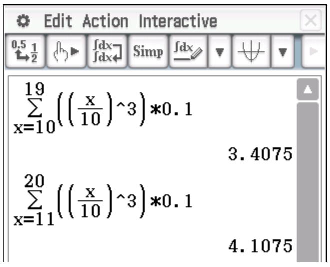
$$= 6\frac{3}{4} \quad \text{or} \quad 6.75 \text{ units}^2$$

## Question 16

Underestimation

$$0.1[1^3 + 1.1^3 + 1.2^3 + \ldots 1.9^3]$$
$$= 3.4075$$




Overestimation

$$0.1[1.1^3 + 1.2^3 + \ldots 2^3]$$
$$= 4.1075$$

$$\bar{x} = \frac{3.4075 + 4.1075}{2}$$
$$= 3.7575 \text{ units}^2$$

## Question 17

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi \times 5^3$$
$$= 524 \text{ cm}^3 \text{ (nearest)}$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{\delta V}{\delta r} \approx 4\pi r^2$$
$$\partial V \approx 4\pi r^2 dr$$
$$\approx 4\pi \times 5^2 \times 0.1$$
$$\approx 31.4$$
$$\therefore V \text{ cm}^3 \pm b \text{ cm}^3$$
$$= 524 \text{ cm}^3 \pm 31 \text{ cm}^3$$